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# $P_\lambda$ -FILTERS AND REGULAR EMBEDDINGS OF BOOLEAN ALGEBRAS

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**ABSTRACT.** This is a résumé of previously published results on an analogue of Mathias forcing associated to the tree  $\omega^{<\omega}$  endowed with a topology appointed by a set of filters.

## 1. INTRODUCTION

We shall consider the set

$$\text{Seq} = \omega^{<\omega} = \bigcup \{ {}^n\omega : n < \omega \}$$

of all finite sequences of natural numbers. The set  $\text{Seq}$  is a tree with a natural order defined as follows: for  $s, t \in \text{Seq}$  we set

$$s \leq t \iff t \restriction \text{dom}(s) = s.$$

The set of all immediate successors of an element  $s \in \text{Seq}$  is denoted by

$$\text{succ}(s) = \{ t \in \text{Seq} : t \text{ is minimal in } \{ t \in \text{Seq} : t > s \} \}.$$

Hence  $\text{succ}(s) = \{ s \frown n : n \in \omega \}$ , where  $s \frown k$  denotes the concatenation of sequence  $s \in {}^n\omega$  by a number  $k$ , i.e.  $s \frown k = s \cup \{(n, k)\}$  is the sequence that extends  $s$  and whose last term is  $k$ .

**Definition 1.** Assume that  $\mathfrak{F} = (\mathcal{F}_t : t \in \text{Seq})$ , where  $\mathcal{F}_t \subseteq \mathcal{P}(\text{succ}(t))$  is a collection of free filters. A set  $U \subseteq \text{Seq}$  is open in the  $\mathfrak{F}$ -topology on  $\text{Seq}$  whenever

$$(\forall s \in U)(\exists F \in \mathcal{F}_s)(F \subseteq U).$$

Then  $\text{Seq}(\mathfrak{F})$  denotes the set  $\text{Seq}$  endowed with the  $\mathfrak{F}$ -topology.

The idea of  $\mathfrak{F}$ -topology on  $\text{Seq}$  has been given by Szymański [10] and, independently, by Trnkova [11]. Later on it was developed by several other authors. A review of  $\mathfrak{F}$ -topologies and their generalizations can be found in [2]. In this paper, in particular, one can find a proof of the following theorem:

**Theorem 1.** For every  $\mathfrak{F} = (\mathcal{F}_t : t \in \text{Seq})$  the space  $\text{Seq}(\mathfrak{F})$  is a zero-dimensional, nowhere compact Hausdorff space. Moreover,  $\text{Seq}(\mathfrak{F})$  is extremally disconnected iff all the filters in  $\mathfrak{F}$  are ultrafilters.

Here, nowhere compact means that every compact subset of the space is nowhere dense. In the sequel we shall discuss the Boolean algebra  $\text{Clop}(\text{Seq}(\mathfrak{F}))$  of all the clopen (=closed and open) subsets of the space  $\text{Seq}(\mathfrak{F})$ . Clearly,  $\text{Clop}(\text{Seq}(\mathfrak{F}))$  is in fact a field of subsets of  $\text{Seq}$ . Theorem 1 immediately implies the following:

**Corollary 1.** The Boolean algebra  $\text{Clop}(\text{Seq}(\mathfrak{F}))$  is complete iff all the filters in  $\mathfrak{F}$  are ultrafilters.

The space  $\text{Seq}(\mathfrak{F})$  was used in a construction of a complete rigid Boolean algebra. Namely, it was proved by Dow, Gubi and Szymański [7] that if  $\mathfrak{F}$  consists of one weak  $P$ -ultrafilter, then the Boolean algebra  $\text{Clop}(\text{Seq}(\mathfrak{F}))$  is complete and rigid.

Let us recollect some well-known cardinal numbers connected to the set of all the functions from  $\omega$  to  $\omega$  ordered by the relation  $\leq^*$  defined as follows:

$$f \leq^* g \iff (\exists n < \omega)(\forall k > n)(f(k) \leq g(k)).$$

This relation leads to the following cardinal characteristics:

1. The *dominating number*  $\mathfrak{d}$  is defined as follows:

$$\mathfrak{d} = \min\{|D| : (\forall f \in {}^\omega\omega)(\exists g \in D)(f \leq^* g)\}.$$

2. The *boundedness*  $\mathfrak{b}$  denotes the minimal cardinality of an unbounded subset of  ${}^\omega\omega$ , i.e.

$$\mathfrak{b} = \min\{|U| : (\forall f \in {}^\omega\omega)(\exists g \in U)(|\{n \in \omega : f(n) < g(n)\}| \geq \omega)\}.$$

3. The *cardinal number*  $\mathfrak{p}$  is defined as

$$\mathfrak{p} = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \text{Clop}(\mathcal{P}(\omega^*)) \text{ is centered and } \text{Int} \bigcap \mathcal{U} = \emptyset\}.$$

It is well-known that:

$$\omega < \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^\omega.$$

The character of a (free) filter denotes the character of the corresponding subset of  $\omega^*$ , i.e. for every (free) filter  $\mathcal{F} \subseteq \mathcal{P}(\omega)$  there is

$$\chi(\mathcal{F}) = \chi(A_{\mathcal{F}}, \omega^*),$$

where  $A_{\mathcal{F}} = \bigcap \{\text{cl}_{\beta\mathbb{N}} U : U \in \mathcal{F}\}$ .

Using the cardinal  $\mathfrak{d}$  one can calculate the character of a space  $\text{Seq}(\mathfrak{F})$  at every point of  $\text{Seq}$ .

**Proposition 1.** *For every  $\mathfrak{F} = (\mathcal{F}_t : t \in \text{Seq})$  and every  $s \in \text{Seq}$  we have*

$$\chi(s, \text{Seq}(\mathfrak{F})) = \mathfrak{d} + \chi(\mathcal{F}_s).$$

## 2. $P_\lambda$ -FILTERS AND $P_\lambda$ -SETS

Since for every  $s \in \text{Seq}$  the set  $\text{succ}(s)$  of all the successors of  $s$  is countable, in the definition of  $\mathfrak{F}$ -topology on  $\text{Seq}$  instead of filters on  $\mathcal{P}(\text{succ}(s))$  one can consider filters on  $\mathcal{P}(\omega)$ . Let us recall that all filters considered here are assumed to be free filters.

**Definition 2.** *A filter  $\mathcal{F} \subseteq \mathcal{P}(\omega)$  is called a  $P_\lambda$ -filter whenever for every family  $\mathcal{R} \subseteq \mathcal{F}$  of size less than  $\lambda$  there exists  $F \in \mathcal{F}$  such that  $F \subseteq^* U$  for every  $U \in \mathcal{R}$ . A  $P_{\omega_1}$ -filter is simply called a  $P$ -filter.*

Here, as usual,  $F \subseteq^* U$  means that the set  $F \setminus U$  is finite. In the virtue of the well-known result of Shelah,  $P$ -ultrafilters do not exist in ZFC. However, it is quite easy to construct a  $P$ -filter. In fact, if  $\{U_n : n \in \omega\}$  consists of infinite subsets of  $\omega$  and  $U_{n+1} \subseteq^* U_n$  for every  $n \in \omega$ , then there exists an infinite set  $V \subseteq \omega$  such that  $V \subseteq^* U_n$  for every  $n < \omega$ . Hence, by transfinite induction one can construct a sequence  $\{U_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}(\omega)$  of infinite sets such that  $U_\alpha \subseteq^* U_\beta$  for all  $\beta < \alpha < \omega_1$ . Clearly, the family  $\{U_\alpha : \alpha < \omega_1\}$  generates a  $P$ -filter.

For topological spaces we have an analogous definition.

**Definition 3.** *For  $\lambda \geq \omega$  and a topological space  $X$ , a set  $S \subseteq X$  is called a  $P_\lambda$ -set provided that  $S$  is contained in the interior of the intersection of every family of size less than  $\lambda$  consisting of open neighborhoods of  $S$ . Also, a  $P$ -set is just a  $P_{\omega_1}$ -set.*

Let us note that if a Tychonoff space  $X$  is nowhere compact then  $X$  is simultaneously dense and boundary in the Čech–Stone compactification  $\beta X$ . Hence, by Theorem 1,  $\text{Seq}(\mathfrak{F})$  is dense and boundary in  $\beta \text{Seq}(\mathfrak{F})$ . The next theorem has been recently proved in [3].

**Theorem 2** (Błaszczyk and Brzeska [3]). *If  $\mathfrak{F} = (\mathcal{F}_s : s \in \text{Seq})$  is a collection of  $P_\lambda$ -filters and  $\omega < \lambda \leq \mathfrak{b}$  then the space  $\text{Seq}(\mathfrak{F})$  is a  $P_\lambda$ -set in  $\beta \text{Seq}(\mathfrak{F})$ .*

From Theorem 2 we immediately obtain the following:

**Corollary 2** (Simon [9]). *If every filter in  $\mathfrak{F} = (\mathcal{F}_s : s \in \text{Seq})$  is a  $P$ -filter, then  $\text{Seq}(\mathfrak{F})$  is a  $P$ -set in  $\beta \text{Seq}(\mathfrak{F})$ .*

**Corollary 3** (Juhász and Szymański [8]). *If  $\omega < \lambda \leq \mathfrak{b}$  and  $\mathcal{F}_s = \mathcal{F}$  for every  $s \in \text{Seq}$ , where  $\mathcal{F}$  is a  $P_\lambda$ -ultrafilter, then  $\text{Seq}(\mathfrak{F})$  is a  $P_\lambda$ -set in  $\beta \text{Seq}(\mathfrak{F})$ .*

The theorem of Simon answers a question of Arhangel'skiĭ whereas Juhász–Szymański's theorem leads to some constructions in the theory of calibers and tightness in compact spaces.

### 3. REGULAR EMBEDDINGS

In this section we shall outline some further applications of Theorem 2 in the theory of Boolean algebras. First, we shall recall some definitions. Let  $\mathbb{B}$  be a Boolean algebra. A subalgebra  $\mathbb{A} \subseteq \mathbb{B}$  is *regular* if for every set  $X \subseteq \mathbb{A}$  there is

$$\sup_{\mathbb{A}} X = 1 \implies \sup_{\mathbb{B}} X = 1.$$

A Boolean algebra  $\mathbb{A}$  is *regularly embedded* in a Boolean algebra  $\mathbb{B}$  provided that there exists a monomorphism of  $\mathbb{A}$  into  $\mathbb{B}$  such that the image of  $\mathbb{A}$  is a regular subalgebra of  $\mathbb{B}$ .

The symbol  ${}^\omega 2$  in the next definition, given by Baumgartner [1], denotes the Cantor set.

**Definition 4.** *A filter  $\mathfrak{F} \subseteq \mathcal{P}(\omega)$  is nowhere dense if for every  $f: \omega \rightarrow {}^\omega 2$  there exists a set  $A \in \mathfrak{F}$  such that  $f[A]$  is a nowhere dense subset of the Cantor set.*

The next theorem gives us an unexpected interrelation between nowhere dense ultrafilters and Boolean algebras. Let us recall that a Boolean algebra  $\mathbb{B}$  is  $\sigma$ -centered if it is the union of countably many ultrafilters. An element  $b \in \mathbb{B} \setminus \{0\}$  is an *atom* if there is no  $a \in \mathbb{B}$  such that  $0 < a < b$ . A Boolean algebra is *atomless* if it has no atoms and it is *atomic* whenever below every element of  $\mathbb{B} \setminus \{0\}$  there is an atom.

**Theorem 3** (Błaszczyk and Shelah [4]). *There exists a  $\sigma$ -centered, atomless, complete Boolean algebra which does not contain any atomless, countable, regular subalgebra iff there exists a nowhere dense ultrafilter.*

Easier part of Theorem 3 can be derived from the next theorem. A proof of this theorem can be obtained by some modification of Theorem 17 from [5].

**Theorem 4.** *Assume that  $\mathfrak{F} = (\mathcal{F}_s: s \in \text{Seq})$  is a collection of filters. If  $\mathcal{F}_s = \mathcal{F}$  for every  $s \in \text{Seq}$ , then the countable free algebra  $\text{Fr}(\omega)$  can be embedded in  $\text{Clop}(\text{Seq}(\mathfrak{F}))$  as a regular subalgebra iff the filter  $\mathcal{F}$  is not nowhere dense.*

The next theorem says that from the point of view of regular embeddings atomlessness is a very essential requirement. For a Boolean algebra  $\mathbb{B}$ ,  $\pi(\mathbb{B})$  denotes the minimal size of a dense subset of  $\mathbb{B}$ .

**Theorem 5.** *Assume that  $\mathfrak{F} = (\mathcal{F}_s: s \in \text{Seq})$  is a collection of  $P_\lambda$ -filters where  $\omega < \lambda \leq \mathfrak{b}$ . If  $\mathbb{B} \subseteq \text{Clop}(\text{Seq}(\mathfrak{F}))$  is a regular subalgebra and  $\pi(\mathbb{B}) < \lambda$ , then  $\mathbb{B}$  is an atomic algebra.*

A topological version of this theorem has been proved in [3]; see Theorem 3.6.

### 4. SKELETAL MAPPINGS

A continuous mapping  $f: X \rightarrow Y$  is called *skeletal* whenever for every open and dense set  $G \subseteq Y$  the set  $f^{-1}[G]$  is dense in  $X$ . Skeletal maps are also known as *semi-open* mappings. The mapping  $f$  is semi-open whenever for every non-empty open set  $U \subseteq X$ , the image  $f[U]$  has a non-empty interior. It appears that skeletal mappings of topological spaces correspond to regular embeddings of Boolean algebras.

**Proposition 2.** *An embedding of Boolean algebras is regular iff the continuous surjection of the corresponding mapping of Stone spaces is skeletal.*

The following example shows that skeletal mappings are easy to construct.

**Example 1.** If  $\mathcal{U}$  is an infinite, maximal disjoint family of clopen subsets of a zero-dimensional compact space  $X$ , then the quotient mapping determined by the closed partition  $\{\{U\}: U \in \mathcal{U}\} \cup \{X \setminus \bigcup \mathcal{U}\}$  is a skeletal mapping onto the one-point compactification of the discrete space of cardinality  $|\mathcal{U}|$ .

Clearly, if a continuous surjection is skeletal and the set of values is dense in itself, then the domain has to be dense in itself as well. However, the above example shows that a skeletal surjection can map a dense in itself compact space onto a space with a dense set of isolated points. The next theorem, proved in [3] says a bit more about it.

**Theorem 6.** Assume that  $\mathfrak{F} = (\mathcal{F}_s: s \in \text{Seq})$  is a collection of  $P_\lambda$ -filters where  $\omega < \lambda \leq \mathfrak{b}$ . If  $X$  is a Hausdorff space with  $\pi w(X) < \lambda$  and a continuous surjection  $f: \beta \text{Seq}(\mathfrak{F}) \rightarrow X$  is skeletal then the set of isolated points in  $X$  is dense.

In connection to skeletal mappings Burke [6] has introduced the following notion:

**Definition 5.** A continuous mapping  $f: X \rightarrow Y$  is called nowhere constant if  $f^{-1}(y)$  is nowhere dense for every  $y \in Y$ .

Clearly, if  $Y$  is a  $T_1$ -space, a mapping  $f: X \rightarrow Y$  is skeletal and the set  $f[X]$  is dense in itself, then  $f$  is nowhere constant. Example 1 shows that in general the converse is not true: skeletal mappings do not have to be nowhere constant. However, the following theorem of Burke shows an interesting connection between nowhere constant and skeletal mappings.

**Theorem 7** (Burke [6]). If  $X$  is Tychonoff and there is a nowhere constant continuous function from  $X$  into  $\mathbb{R}$ , and  $\pi w(X) < \mathfrak{p}$ , then there also exists a skeletal function from  $X$  into  $\mathbb{R}$ .

In particular, the above theorem shows that if a compact metric space has a nowhere constant mapping into the reals, then it has also a skeletal mapping into reals. Also, Burke [2] asked whether there exists (in ZFC) a Tychonoff space of  $\pi$ -weight  $\mathfrak{p}$  which has a nowhere constant mapping onto  $\mathbb{R}$  but does not have a skeletal mapping onto  $\mathbb{R}$ . We give a partial answer to this question.

**Theorem 8.** If  $\mathfrak{F} = (\mathcal{F}_s: s \in \text{Seq})$  is a collection of  $P$ -filters of character  $\aleph_1$ , then the space  $\text{Seq}(\mathfrak{F})$  is a space of the  $\pi$ -weight equal to  $\mathfrak{d}$  which has a nowhere constant mapping into  $\mathbb{R}$  but does not have a skeletal mapping into  $\mathbb{R}$ .

The above theorem also shows that nowhere constant mapping does not have to be skeletal.

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